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NUMERICAL SOLUTION OF STEFAN PROBLEMS

A. B. CROWLEY*
Department of Mathematics, Brunel University, Uxbridge, Middx., U.K.

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Abstract—The extension of the enthalpy method to multi-dimensional Stefan problems is outlined. The method is applied to the numerical solution of a problem involving the solidification of a square cylinder of fluid when the surface temperature is lowered at a constant rate. One aim of this paper is to demonstrate that the numerical method successfully predicts the experimental results which have been published for this problem. The same technique is then applied to a similar problem, in which the surface temperature is lowered discontinuously at the initial instant, and the results compared with those obtained by other authors.

NOMENCLATURE

c, specific heat;

H, dimensionless enthalpy, $\left(=\dfrac{H^*-H_F^*}{c(T_F^*-T_0^*)}\right)$;

H^*, enthalpy;

H_F^*, enthalpy of solid phase at T_F^*;

l, half-width of cylinder;

L, dimensionless latent heat, $\left(=\dfrac{L^*}{c(T_F^*-T_0^*)}\right)$;

L^*, latent heat;

$S(x,y,t)=0$, freezing front in two-dimensional problem;

$s(t)$, position of freezing front in one-dimensional problem;

t, dimensionless time or Fourier number, $=\left(\dfrac{t^*\kappa}{\rho c l^2}\right)$;

t^*, time;

T, dimensionless temperature, $=\left(\dfrac{T^*-T_F^*}{T_F^*-T_0^*}\right)$;

T^*, temperature;

T_F^*, freezing temperature;

T_0^*, surface temperature of cylinder;

x, y, cartesian coordinates;

κ, thermal conductivity;

ρ, density.

Subscripts

L, liquid phase;

s, solid phase;

1. INTRODUCTION

STEFAN problems, that is problems of heat conduction with change of phase, have attracted much interest in the literature, and recently attention has been centred on the solution of multi-dimensional problems. As no exact analytical solutions of such problems are available, we must in general resort to numerical methods, several of which have been proposed [1–3]. In this situation the experimental results of Saitoh [4], for the two-dimensional freezing of water, are of particular value, as they provide, for the first time, the opportunity for a direct comparison between numerical results and experimental data.

The numerical method proposed in this paper is the straightforward extension to higher dimensions of the enthalpy method, described in one dimension by Kamenomostskaja [5] and Atthey [6]. The latter applied the method to a one-dimensional spot-welding problem. The advantages of the enthalpy reformulation are that the problem to be solved is formulated in a fixed region, and no modification of the numerical scheme is necessary in order to satisfy the conditions at the moving phase-change interface. Although the interface is not tracked explicitly, its position may be found by extrapolation if this is desired. For the sake of simplicity and in order to allow comparison with both other numerical methods and experimental data, the calculations in this paper are for somewhat academic one-phase problems in two dimensions. However the enthalpy method is immediately applicable to two-phase problems, the appearance or disappearance of a phase causing no complications, and to fully three-dimensional situations. Thus this method may be used in a variety of engineering fields where Stefan problems arise, such as the solidification and corrosion of metals, welding processes and the diffusion-controlled growth of crystals in solution. Several of these examples are described in [7].

We consider two problems of the inward solidification of a square cylinder of liquid, initially at its freezing temperature, with different surface conditions. The first of these, in which the surface temperature is lowered at a constant rate, corresponds to the experimental configuration employed by Saitoh. The second problem, in which the surface temperature is lowered discontinuously at the initial instant, is one for which several sets of numerical results have previously been published. The enthalpy method is first applied to the solution of Saitoh's problem, and good agreement between the numerical results and experimental data is obtained. The method is then applied to the second problem, and the results compared with those previously published.

*Present address: Oxford University Computing Laboratory, 19, Parks Road, Oxford OX1 3PJ, U.K.

2. THE PHYSICAL PROBLEM AND THE ENTHALPY REFORMULATION

We consider an infinitely long cylinder of square cross-section, filled with liquid at its freezing temperature. The lowering of the surface temperature causes a frozen layer to grow inwards from the surface. In this model the density, specific heat and thermal conductivity of the solid are taken as constant, and it is assumed that there are no density changes on freezing. Hence there is no convection in the liquid region.

In non-dimensional variables, the cylinder's cross-section may be taken as $(-1 \leqq x \leqq 1, -1 \leqq y \leqq 1)$ and the time scale chosen is that of conduction, (i.e. $t = t^*\kappa/\rho c l^2$), while the freezing temperature is chosen as the origin of the non-dimensional temperature scale. Let D be the domain, initially zero, occupied by the frozen material and let the freezing front be $S(x, y, t) = 0$ (see Fig. 1).

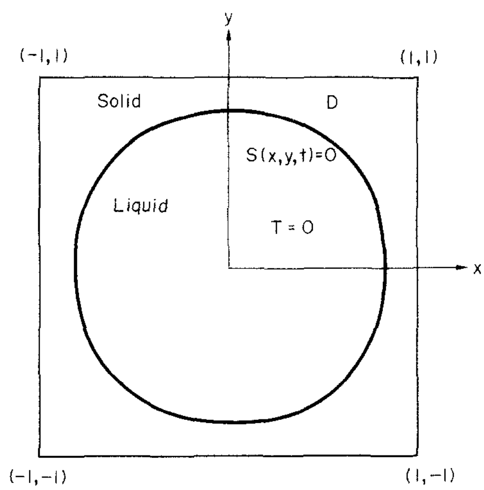

FIG. 1. Sketch of the x, y plane.

A suitable non-dimensional model is thus

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \quad \text{in } D, \tag{2.1}$$

with

$$T \equiv 0 \tag{2.2}$$

in the liquid region, where T denotes the temperature and t the time. At the freezing front, $S(x, y, t) = 0$,

$$T = 0, \tag{2.3}$$

and by conservation of energy

$$L\frac{\partial S}{\partial t} = -(\nabla T \cdot \nabla S)|_s, \tag{2.4}$$

where L denotes the non-dimensional latent heat, and $(\nabla T \cdot \nabla S)|_s$ is the limit of $(\nabla T \cdot \nabla S)$ as the freezing front is approached in D.

In the first problem the surface temperature is lowered at a constant rate, so the boundary condition at the fixed surface is

$$T = -kt \quad \text{on} \quad (x^2-1)(y^2-1) = 0 \tag{2.5}$$

where $k > 0$. In the second problem, the temperature is lowered discontinuously, and then maintained constant, so the appropriate condition is

$$T = -1 \quad \text{on} \quad (x^2-1)(y^2-1) = 0, \quad t > 0. \tag{2.6}$$

(2.1)–(2.5), (2.1)–(2.4) and (2.6) constitute complete descriptions of the two problems to be solved. We next outline the enthalpy reformulation of these problems, and the resulting numerical method.

We introduce the enthalpy, H, in order to reformulate the problem on the fixed domain $(-1 \leqq x \leqq 1, -1 \leqq y \leqq 1)$. In non-dimensional variables the temperature is defined in terms of the enthalpy by

$$T = \begin{cases} H & H \leqq 0, \\ 0 & 0 < H < L, \\ H - L & H \geqq L, \end{cases} \tag{2.7}$$

where $T = 0$, $H = L$ corresponds to liquid at its freezing temperature. (2.1), (2.2) then become

$$\frac{\partial H}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{2.8}$$

where H is continuous. Across the discontinuity in H, which occurs at the freezing front due to the liberation of latent heat, the conservation form of (2.8) yields

$$(H|_L - H|_s)\frac{\partial S}{\partial t} = L\frac{\partial S}{\partial t} = (\nabla T \cdot \nabla S)|_L - (\nabla T \cdot \nabla S)|_s$$

which reduces immediately to (2.4) since in this case $T \equiv 0$ in the liquid. Thus when the conduction equation is written in the form (2.8), with $T = T(H)$ defined by (2.7), the conditions (2.2), (2.3) are automatically satisfied by any solution of the weak form of (2.8).

Oleinik [8] showed that the problem (2.7), (2.8) with appropriate boundary and initial conditions has a unique weak solution, and it may also be shown [5, 6] that an explicit finite difference converges to this solution. Jerome [9] has shown that an implicit finite difference scheme converges to the weak solution.

Thus in order to find a numerical solution, we solve the finite difference forms of (2.7), (2.8) together with the initial condition $T = 0$, $H = L$ in $-1 < x < 1, -1 < y < 1$, and the appropriate boundary condition (2.5) or (2.6).

By symmetry it is sufficient to work on the first quadrant of the square, with the boundary conditions $\partial T/\partial x = 0$ on $x = 0$, $\partial T/\partial y = 0$ on $y = 0$.

3. RESULTS AND DISCUSSION

All the numerical results for the first problem were calculated using an explicit finite difference scheme to solve (2.7), (2.8) with the appropriate boundary conditions. The calculations using a 20×20 spatial mesh, took 185 s of computing time on a CDC 7600. Figure 2 shows the position of the freezing front on the axis, and its x coordinate on the diagonal plotted against time. We see that the numerical results agree well with the experimental data, both on the axis and along the diagonal.

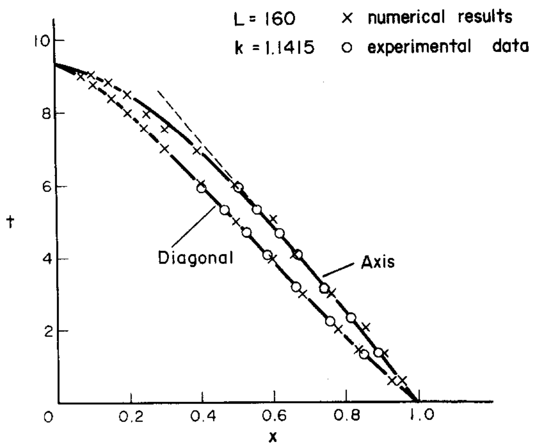

FIG. 2. The x coordinate of the freezing front on the axis and on the diagonal for Saitoh's problem. The dotted line shows the position of the front in the analogous one dimensional problem (3.1).

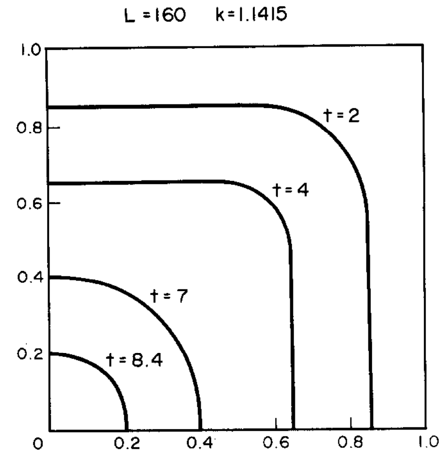

FIG. 3. Position of the freezing front at various times.

During an initial time interval in which, away from the corners, the freezing front remains parallel to the fixed surface, heat conduction parallel to the boundaries is negligible. Thus during this interval the solution of the two-dimensional problem near the axis is well approximated by the solution of the corresponding one-dimensional problem. The exact solution of the one-dimensional analogue of Saitoh's problem is not known, but in this case L is large (~ 160). Thus we may use a perturbation solution in inverse powers of L, which yields for the position, $x = s(t)$, of the freezing front

$$s = 1 - t\left(\frac{k}{L}\right)^{1/2} + \frac{t^2}{6}\left(\frac{k}{L}\right)^{3/2} + \ldots \qquad (3.1)$$

We have therefore plotted this solution for comparison with the solution of the two-dimensional problem near the axis, and the agreement is excellent until $t \sim 6$, when the front has moved halfway to the centre. Figure 3 shows the freezing front at various times.

Figures 4 and 5 show the position of the freezing front on the axis and its x coordinate on the diagonal as functions of time for the second problem, as calculated by the enthalpy method. Also shown are the results obtained by Allen and Severn [1], Lazaridis [2], Crank and Gupta [3]. Here the exact one-dimensional solution for the position $s(t)$ of the freezing front on the axis, $s(t) = 1 - \alpha\sqrt{t}$, [10], where α satisfies

$$L\frac{\sqrt{\pi}}{2}\alpha \exp\frac{\alpha^2}{4} \operatorname{erf}\frac{\alpha}{2} = 1,$$

is plotted in Fig. 4 for comparison. The numerical solution for solidification in an infinite corner obtained by Rathjen and Jiji [11] is also shown in Fig. 5 since this is a good approximation to the solution of the present problem while the front remains parallel to the fixed surface away from the corners.

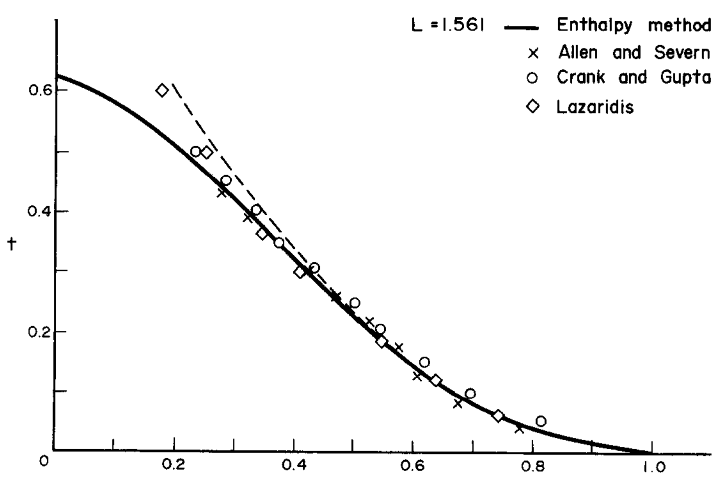

FIG. 4. The x coordinate of the freezing front on the axis for the second problem. The solid line shows the results obtained from the enthalpy method, the broken line the front $s(t) = 1 - 1.034\sqrt{t}$, for the corresponding one dimensional problem.

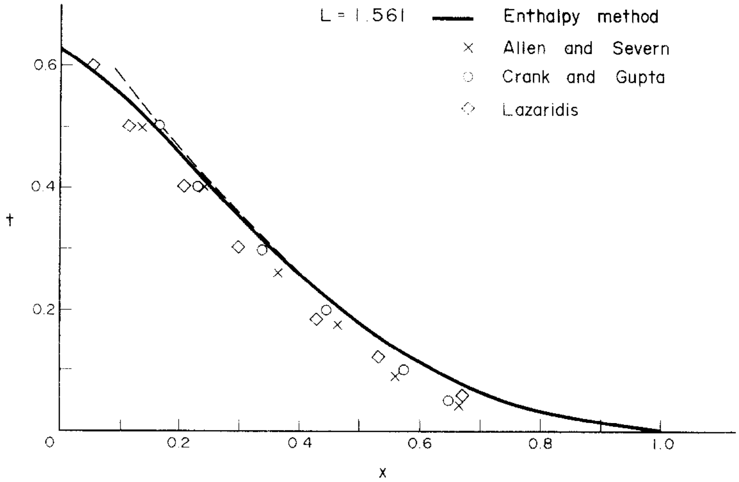

FIG. 5. The x coordinate of the freezing front on the diagonal. Again the solid line shows the results obtained by the enthalpy method, while the broken line shows the solution for solidification in an infinite corner [11].

For this problem an explicit scheme on a 40×40 spatial mesh was used to obtain the results shown, in 210 s of computing time. However, using a 20×20 spatial mesh and 18 s computing time, very similar curves were obtained, with a difference of only 3% in the time of final solidification ($t \sim 0.625$) estimated from these curves. Implicit schemes were also used for the second problem, and the results were in excellent agreement with those on the same spatial mesh, but less economical on computing time.

The freezing front as calculated by the enthalpy method is squarer than those from the other numerical schemes. However, as the numerical results for Saitoh's problem are a good fit to the experimental data on both the axis and the diagonal, it would appear that this shape is correct. This squarer shape of the freezing front is also in excellent agreement with the numerical solution for the solidification of an infinite corner [11].

We conclude that the enthalpy reformulation provides an accurate and reliable scheme for the numerical solution of Stefan problems. Moreover, the method is computationally far simpler than that for the original equations proposed in [2], and is equally suitable for heat flux boundary conditions at the fixed surface, unlike the technique described in [3]. This successful comparison with experiment should give confidence in the mathematical treatment, and encourage the acceptance of the method as a useful tool for exploring practical problems.

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RESOLUTION NUMERIQUE DES PROBLEMES DE STEFAN

Résumé—On traite de l'extension de la méthode de l'enthalpie aux problèmes multidimensionnels de Stefan. La méthode est appliquée à la solution numérique d'un problème qui concerne la solidification d'un cylindre à base carrée de fluide lorsque la température de surface est abaissée à vitesse constante. Un but de cet article est de montrer que la méthode numérique prédit avec succès les résultats expérimentaux publiés sur ce problème. La même technique est aussi appliquée à un problème semblable dans lequel la température de surface est abaissée de façon discontinue à l'instant initial et les résultats sont comparés à ceux obtenus par d'autres auteurs.

NUMERISCHE LÖSUNG VON STEFAN-PROBLEMEN

Zusammenfassung—Die Erweiterung der Enthalpiemethode auf mehrdimensionale Stefan-Probleme wird umrissen. Die Methode wird auf die numerische Lösung eines Problems angewandt, welches die Erstarrung einer Flüssigkeit in einem quadratischen Zylinder bei konstanter Absenkung der Oberflächentemperatur behandelt. Ein Ziel dieser Arbeit ist es, zu zeigen, daß die numerische Methode mit Erfolg die experimentellen Ergebnisse beschreiben kann, die für das genannte Problem veröffentlicht wurden. Die gleiche Technik wird dann auf ein ähnliches Problem angewandt, bei welchem die Oberflächentemperatur während des ersten Augenblickes diskontinuierlich gesenkt wird. Die Ergebnisse werden mit denen anderer Autoren verglichen.

ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧ СТЕФАНА

Аннотация — Рассматривается возможность применения метода энтальпии для решения многомерных задач Стефана. Метод используется для численного решения задачи в случае затвердевания жидкости в форме цилиндра с квадратным сечением при понижении температуры поверхности с постоянной скоростью. Работа ставила своей целью показать, что численный метод успешно предсказывает экспериментальные результаты, ранее опубликованные по данной задаче. Затем этот метод был применен для аналогичной задачи, когда температура поверхности понижалась скачкообразно в начальный момент времени. Полученные результаты сравнивались с данными других авторов.